Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 21 (2008) 554–557
Applied Mathematics Letters
www.elsevier.com/locate/aml

An extension of the mountain pass lemma

Lizhou Wang*, Dongsheng Li

Department of Mathematics, College of Science, Xi'an Jiaotong University, Xi'an 710049, PR China

Received 7 July 2005; received in revised form 7 May 2007; accepted 2 July 2007

Abstract

We present a more general form of the mountain pass lemma. It asserts that a C^1 functional which satisfies the Palais–Smale condition admits a critical value when the connectedness of certain level sets changes. We also give an improved form of a theorem given in [A. Bahri, H. Berestycki, A perturbation method in critical point theory and applications, Trans. Amer. Math. Soc. 267 (1) (1981) 1–32], which characterizes the existence of the critical value by means of contractibility properties of the level sets.

Keywords: Mountain pass lemma; Palais–Smale condition; Connectedness; Contractibility; Level set

1. Introduction and main results

The mountain pass lemma is one of the most elementary theorems in the modern variational approach to nonlinear problems. It admits several variants and extensions. For this aspect, we refer the reader to the survey papers [2–4] and the monograph [5] and the references therein. In this work, we present a more general form of the mountain pass lemma, which characterizes the existence of the critical value by the change of a basic topological property—connectedness of certain level sets. We also wish to point out an improved form of a theorem in [1], which describes the existence of the critical value by considering another topological property—the contractibility property of the level sets.

First, we introduce some notation. Let E be a Banach space. We denote by $B_r(u)$ the open ball centered at $u \in E$ with radius $r > 0$, $\overline{B}_r(u)$ its closure and $\partial B_r(u)$ its boundary. Let $I \in C^1(E, R)$. We introduce the level set

$$I_c = \{u \in E; I(u) \le c\}, \quad c \in R.$$

We say that I satisfies the Palais–Smale condition (henceforth denoted by (PS)) if any sequence $\{u_m\} \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence. The mountain pass lemma reads as

Theorem 1. *Let E be a Banach space and $I \in C^1(E, R)$. Suppose I satisfies (PS) and there exist $u_0, u_1 \in E$, $\rho > 0$ such that*
(I_1) $u_1 \overline{\in} \overline{B}_\rho(u_0)$;
(I_2) $\max\{I(u_0), I(u_1)\} < \inf_{u \in \partial B_\rho(u_0)} I(u)$.

* Corresponding author.
E-mail addresses: wangliz@mail.xjtu.edu.cn (L. Wang), lidsh@mail.xjtu.edu.cn (D. Li).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.07.005

Then, I possesses a critical value which can be characterized as

$$c = \inf_{A \in \Gamma} \max_{u \in A} I(u)$$

where

$$\Gamma = \{\gamma([0, 1]); \gamma \in C([0, 1], E), \gamma(0) = u_0, \gamma(1) = u_1\}.$$

In this work, we will prove the following more general form of the mountain pass lemma:

Theorem 2. *Let E be a Banach space and $I \in C^1(E, R)$. Suppose I satisfies (PS) and there exist $u_0, u_1 \in E, a, b \in R$, such that*
(I_3) *u_0, u_1 lie in different components of I_a;*
(I_4) *u_0, u_1 lie in the same component of I_b.*
Then, I possesses a critical value in $[a, b]$ which can be characterized as
$c = \inf\{c' \in (a, b]; u_0, u_1$ *lie in the same component of* $I_{c'}\}$.

Roughly speaking, Theorem 2 asserts that a functional $I \in C^1(E, R)$ which satisfies a certain compactness condition assumes a critical value when the connectedness of the level set I_c changes. Suppose that I satisfies the conditions in Theorem 1. Set

$$a = \frac{1}{2}(\max\{I(u_0), I(u_1)\} + \inf_{u \in \partial B_\rho(u_0)} I(u)), \qquad b = \max_{t \in [0,1]} I(tu_0 + (1 - t)u_1).$$

Then, (I_1)(I_2) imply that $u_0, u_1 \in I_a$ and u_0, u_1 lie in different components of I_a. Clearly, u_0, u_1 lie in the same component of I_b by definition. Thus, by Theorem 2, I admits a critical value. This observation shows that Theorem 2 is a more general form of the mountain pass lemma.

Noticing that the Banach space E itself is connected, we obtain the following simple corollary of Theorem 2:

Corollary. *Let E be a Banach space and $I \in C^1(E, R)$. Suppose I satisfies (PS) and there exist $u_0, u_1 \in E, a \in R$, such that*
(I_3) *u_0, u_1 lie in different components of I_a.*
Then, I possesses a critical value in $[a, \infty)$ which can be characterized as
$c = \inf\{c' \in (a, \infty); u_0, u_1$ *lie in the same component of* $I_{c'}\}$.

We will prove Theorem 2 in Section 2. The crucial step in the proof is the following lemma:

Lemma. *Let E be a Banach space and $I \in C^1(E, R)$. Suppose I satisfies (PS) and I has no critical value in $[a, b]$. Then, I_a is a retract of I_b.*

This lemma was proved in [1] under the more restrictive condition $I \in C^2(E, R)$. Thus, it is an improvement of the corresponding result in [1]. As a consequence, we can also improve another conclusion in [1] somewhat, which identifies the critical value of a functional by investigating the contractibility property of the level set I_a. In fact, two weaker forms of the following Theorem 3 were proved in [1], either assuming the stronger condition $I \in C^2(E, R)$ to obtain the same conclusion as Theorem 3, or assuming $I \in C^1(E, R)$ to obtain a weaker conclusion.

Theorem 3. *Let E be a Banach space and $I \in C^1(E, R)$. Suppose I satisfies (PS) and there exist $a, b \in R$ such that*
(I_5) *I_a is not contractible to a point in itself;*
(I_6) *I_a is contractible to a point in I_b.*
Then, I possesses a critical value in $[a, b]$ which can be characterized as
$c = \inf\{c' \in (a, b]; I_a$ *is contractible to a point in* $I_{c'}\}$.

Remarks. (i) The smooth assumption in the above theorems, corollary and lemma may be weakened. For example, in Theorem 2, we may assume $I \in C(E, R) \bigcap C^1(I_{b+\epsilon} - I_{a-\epsilon}, R)$ for some $\epsilon > 0$.

(ii) The (PS) in the above results can be replaced by the following local form: I satisfies the (PS) if any sequence $\{u_m\} \subset E$ for which $c \leq I(u_m) \leq C$ for some $C > c$, and $I'(u_m) \to 0$ as $m \to \infty$ is precompact.

(iii) The conclusions in this work hold for a more general setting. For example, the functional I may be defined on a Banach manifold or, more generally, on a complete connected C^1-Finsler manifold [2,4].

2. Proof of the results

In this section, we present the proofs of the conclusions in Section 1. First, we prove the lemma; then, using it, we prove Theorems 2 and 3.

Proof of the lemma. To prove the lemma, we shall construct a map $\gamma \in C(I_b, I_a)$, such that

$$\gamma(u) = u, \quad \forall u \in I_a.$$

This will be realized in two steps. First, we will construct the pseudo-gradient flow $\eta \in C(R \times E, E)$. Then, using the pseudo-gradient flow, we construct the desired map.

The pseudo-gradient flow η will be constructed as the solution of an ordinary differential equation. We only outline the construction here (see [4] for similar result and detailed proofs).

A few preliminaries are needed before setting up this differential equation. By (PS) one can show that there exists $\epsilon > 0$ such that I has not critical value in $[a-\epsilon, b+\epsilon]$. There exists a Lipschitz continuous map ϕ such that

$$0 < \phi(u) < 1, \quad \text{if } I(u) \in (a-\epsilon, a) \cup (b, b+\epsilon)$$
$$\phi(u) = 1, \quad \text{if } I(u) \in [a, b]$$

and

$$\phi(u) = 0, \quad \text{if } I(u) \in (-\infty, a-\epsilon] \cup [b+\epsilon, \infty).$$

We can construct a pseudo-gradient vector field V on $\widetilde{E} = \{u \in E; I'(u) \neq 0\}$ such that V is locally Lipschitz continuous and satisfies

$$\|V(u)\| \le 2\|I'(u)\|, \quad \forall u \in \widetilde{E}, \tag{2.1}$$
$$I'(u)V(u) \ge \|I'(u)\|^2, \quad \forall u \in \widetilde{E}. \tag{2.2}$$

Notice that (2.2) implies

$$\|V(u)\| \ge \|I'(u)\|, \quad \forall u \in \widetilde{E}. \tag{2.3}$$

Next define $h(s) = 1$ if $s \in [0, 1]$ and $h(s) = 1/s$ if $s \ge 1$. Finally set $W(u) = -\phi(u)h(\|V(u)\|)V(u)$ for $u \in \widetilde{E}$ and $W(u) = 0$ otherwise. Then, by construction, W is locally Lipschitz continuous on E and $0 \le \|W\| \le 1$.

Now we can define the map η. Consider the Cauchy problem

$$\frac{d\eta}{dt} = W(\eta), \qquad \eta(0, u) = u.$$

It can be proved that the solution $\eta \in C(R \times E, E)$ (see [2] for detailed proof).

Next we explore the properties of the pseudo-gradient flow η. By definition,

$$\frac{dI(\eta(t, u))}{dt} = I'(\eta(t, u))W(\eta(t, u)) \le 0, \quad \forall u \in E, \tag{2.4}$$

and the strict inequality holds if $I(u) \in (a-\epsilon, b+\epsilon)$. Thus, $I(\eta(t, u))$ is nonincreasing in t, and strictly decreasing if $I(u) \in (a-\epsilon, b+\epsilon)$. By (PS), there exists $\delta \in (0, 1)$, such that

$$\|I'(u)\| \ge \delta, \quad \text{if } I(u) \in [a, b]. \tag{2.5}$$

If $I(u) \in (a-\epsilon, b+\epsilon)$, then $I(\eta(t, u)) \in (a-\epsilon, b+\epsilon)$ since $W(\eta(t, u)) = 0$ if $I(\eta(t, u)) \overline{\in} (a-\epsilon, b+\epsilon)$. We claim that, if $u \in I_b - I_a$, their exists a unique $T(u) > 0$, such that $I(\eta(T(u), u)) = a$. In fact, if $I(u) \in [a, b]$ and for all $t > 0$, $I(\eta(t, u)) \in [a, b]$, then,

$$\begin{aligned} I(\eta(t, u)) &= I(u) + \int_0^t \frac{d}{ds} I(\eta(s, u))ds \\ &= I(u) - \int_0^t I'(\eta(s, u))h(\|V(\eta(s, u))\|)V(\eta(s, u))ds \end{aligned}$$

$$\begin{aligned}
&\leq I(u) - \int_0^t \|I'(\eta(s,u))\|^2 h(\|V(\eta(s,u))\|)\mathrm{d}s \\
&\leq I(u) - \delta \int_0^t \|I'(\eta(s,u))\| h(\|V(\eta(s,u))\|)\mathrm{d}s \\
&\leq I(u) - \delta/2 \int_0^t \|V(\eta(s,u))\| h(\|V(\eta(s,u))\|)\mathrm{d}s \\
&\leq I(u) - \delta^2 t/2, \qquad (2.6)
\end{aligned}$$

where we successively used (2.2), (2.3) and (2.5). Obviously, (2.6) cannot hold for large t. Hence, for each u, such that $I(u) \in [a, b]$, there exists $\widetilde{T}(u) > 0$, such that $I(\eta(\widetilde{T}(u), u)) < a$. Thus by the monotonicity of $I(\eta(t, u))$, we conclude that there exists a unique $T(u) > 0$ such that $I(\eta(T(u), u)) = a$. In fact, $T(u)$ is the unique solution of the equation

$$I(u) - a = \int_0^T I'(\eta(s,u))W(\eta(s,u))\mathrm{d}s, \quad u \in I_b - I_a.$$

By the implicit function theorem, we see that $T \in C(I^{-1}((a, b]), R)$. The estimate (2.6) also implies that

$$T(u) \leq 2(I(u) - a)/\delta^2.$$

Thus, $T(u) \to 0$ as $I(u) \to a$.

Set

$$\gamma(u) = \begin{cases} \eta(T(u), u), & \text{if } I(u) \in (a, b], \\ u, & \text{if } I(u) \leq a. \end{cases}$$

The above arguments on T show that $\gamma \in C(I_b, I_a)$, and $\gamma(u) = u$ if $u \in I_a$ by definition. Hence, γ is the desired retraction. This completes the proof. □

Proof of Theorem 2. We prove by contradiction. Suppose that c is not a critical value of I. We will derive a contradiction.

Obviously, $a \leq c \leq b$. First, we claim that $c > a$. By (PS), there exists $\epsilon > 0$ such that I has no critical value in $[c, c + \epsilon]$. According to the definition of c, there exists $c' \in [c, c + \epsilon]$ such that u_0, u_1 lie in the same component of $I_{c'}$, say $A \subset I_{c'}$. Thus, $A \subset I_{c+\epsilon}$ and is connected. Applying the lemma, we obtain $\gamma \in C(I_{c+\epsilon}, I_c)$ such that $\gamma(u) = u$ if $u \in I_c$. Let $B = \gamma(A)$. Then, $B \subset I_c$, $u_0, u_1 \in B$ and B is connected. This implies that u_0, u_1 lie in the same component of I_c. Thus, $c > a$ by (I_3).

By (PS), there exists another $\epsilon > 0$ such that I has no critical value in $[c - \epsilon, c]$. We may assume $c - \epsilon > a$ since we have shown $c > a$. Applying the lemma again, we obtain $\beta \in C(I_c, I_{c-\epsilon})$ such that $\beta(u) = u$ for $u \in I_{c-\epsilon}$. Let $D = \beta(B)$. Then, $D \subset I_{c-\epsilon}$, $u_0, u_1 \in D$ and D is connected. This implies that u_0, u_1 lie in the same component of $I_{c-\epsilon}$, which contradicts the definition of c. Thus, we conclude that c must be a critical value of I. □

Proof of Theorem 3. First, notice that if I_a is contractible to a point in $A \subset E$, and $B \subset I_a$ is a retract of A, then I_a is also contractible to a point in B. Then, Theorem 3 can be proved in completely the same manner as Theorem 2. We omit the details here. □

References

[1] A. Bahri, H. Berestycki, A perturbation method in critical point theory and applications, Trans. Amer. Math. Soc. 267 (1) (1981) 1–32.
[2] I. Ekeland, N. Ghoussoub, Selected new aspects of the calculus of variations in the large, Bull. A.M.S. 39 (2) (2002) 207–265.
[3] L. Nirenberg, Variational methods in nonlinear problems, in: Lecture Notes in Mathematics, vol. 1365, Springer-Verlag, 1989, pp. 100–119.
[4] P.H. Rabinowitz, Minmax methods in critical point theory with applications to differential equations, in: C.B.M.S., vol. 65, A.M.S., 1986.
[5] M. Struwe, Variational Methods and their Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, 1990.